1 Topology and Functional Spaces

Exercise 1.1. Let (X,μ) be a σ-finite measured space,and that $f:X\to\mathbb{R}$ be a measurable function.

1. Show that
$$\int_X |f|\,d\mu=\int_0^\infty \mu\left(X\cap\{x:|f(x)|>t\}\right)d\mathscr{L}^1(t),$$
where $\mathscr{L}^1$ is the 1-dimensional Lebesgue measure.

2. Deduce that for all $0<p<\infty$, we have
$$\int_X |f|^p d\mu=p\int_0^\infty t^{p-1}\mu\left(X\cap\{x:|f(x)|>t\}\right)d\mathscr{L}^1(t).$$

3. Show that **1**. holds for a general measure, by proving the theorem first in the case of an indicator function, and for positive step functions.

Exercise 1.2. Let $f\in L^1(\mathbb{R}^n)$ be such that
$$\int_{\mathbb{R}^n} f\varphi=0\qquad \text{for all } \varphi\in\mathscr{D}(\mathbb{R}^n).$$
Show that $f=0$.

Exercise 1.3. Let X and Y be two Banach spaces, and $\{T_n\}_{n\in\mathbb{N}}\subset\mathscr{L}(X,Y)$ such that for all $x\in X$, $T_n(x)$ converges as $n\to\infty$ towards a limit denoted by $T(x)$. Show that if $\{x_n\}_{n\in\mathbb{N}}\subset X$ is a sequence such that $x_n\underset{n\to\infty}{\longrightarrow} x\in X$, we have $T_n(x_n)\underset{n\to\infty}{\longrightarrow} T(x)\in Y$.

Exercise 1.4. Let X and Y be two Banach spaces and let $a:X\times Y\to\mathbb{R}$ be a bilinear form satisfying :

1. For all $x\in X$, the map $y\mapsto a(x,y)$ is continuous;
2. For all $y\in Y$, the map $x\mapsto a(x,y)$ is continuous.

Then, show that there exists a constant $C<\infty$ such that
$$|a(x,y)|\leq C\,\|x\|_X\,\|y\|_Y\quad \forall x\in X,\forall y\in Y.$$

Exercise 1.5. Let $a=\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ and $1\leq p\leq\infty$. Assume that for all $x=\{x_n\}_{n\in\mathbb{N}}\subset l^p(\mathbb{N})$, we have
$$\sum_{n\in\mathbb{N}}|a_n||x_n|<\infty.$$

Prove that $a \in l^{p'}(\mathbb{N})$.

We recall that for all $1 \leq q \leq \infty$, we have

$$l^q(\mathbb{N}) = \mathbb{R}^{\mathbb{N}} \cap \left\{ x : \|x\|_{l^q} = \left(\sum_{n \in \mathbb{N}} |x_n|^q \right)^{\frac{1}{q}} < \infty \right\}$$

equipped with its natural norm $\|\cdot\|$ is a Banach space (for $q = \infty$, replace the norm above by the sup norm $\|x\|_{l^\infty} = \sup_{n \in \mathbb{N}} |x_n|$).

Exercise 1.6. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ with $1 < p \leq \infty$ and $f \in L^p(\Omega)$. Show that the following properties are equivalent :

1. $f_n \underset{n\to\infty}{\rightharpoonup} f$ in $\sigma(L^p, L^{p'})$.

2. $\|f_n\|_{\mathrm{L}^p(\Omega)} \leq C$ and $\int_A f_n \, d\mathscr{L}^d \underset{n\to\infty}{\longrightarrow} \int_A f \, d\mathscr{L}^d$ for all $A \subset \Omega$ of finite measure.

If $p = 1$ and $\mathscr{L}^d(\Omega) < \infty$, show that the previous equivalence holds true.

Remark. The equivalence is false in general due to the possibility of mass « escaping » at infinity. The sequence $\{u_n\}_{n\in\mathbb{N}} = \mathbf{1}_{[n,n+1]} \subset L^1(\mathbb{R})$ furnishes a counter-example.

Exercise 1.7. A map $f : X \to Y$ is continuous if and only if it is continuous at every point of X.

From the previous property deduce the Theorem 1.1.11 stated in class.